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SOME MIXED BOUNDARY VALUE PROBLEMS OF ELASTICITY IN A RECTANGULAR DOMAIN†

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Abstract—The general case of a rectangular elastic body in plane strain, when two parallel edges are traction free and mixed conditions are prescribed on the remaining edges, is formulated in terms of a system of two Fredholm integral equations of the second kind. The type of mixed boundary conditions may arise due to compression by smooth punches or some periodic arrangement of a system of cracks in an infinitely long strip. Indentation of a rectangular body by a flat and a parabolic punch is discussed in some detail and numerical results are reported for quantities of practical interests.

1. INTRODUCTION

THE PRESENT study is concerned with an investigation of the two-dimensional plane strain problem of stresses and displacements in a rectangular body. A system of Fredholm integral equations of the second kind is obtained for the general case when two parallel edges are stress free and mixed conditions are prescribed on the remaining edges. The type of mixed boundary conditions considered here may arise due to compression by smooth punches or some periodic arrangement of a system of cracks in an infinite strip. The governing Fredholm equations are derived by extending the study presented in [1] by the present authors in which Papkovich–Fadle eigenfunctions were employed to investigate a certain group of crack configurations in an infinite strip. The integral equations are numerically solved for various cases to obtain quantities of practical interests.

In the past, mixed boundary value problems of elastostatics in rectangular domain have been discussed by Sneddon and Berry [2] for the case when two parallel edges were restrained by smooth rigid planes. On the remaining edges mixed conditions arising due to compression by two smooth punches of identical shape were prescribed. The problem was treated by applying the method of complex variable. It is known, however, that smoothly restrained edge conditions permit treatment of elasticity problems by Fourier series and, therefore, are mathematically straightforward to analyze. On the other hand, more natural boundary conditions of stress free edges introduce considerable analytical difficulties. To handle this situation the method of integral transform [3] has proved quite powerful but its application has been limited to the special cases when the edges are infinitely long. In a series of two papers [4, 5] Keer and Sve have solved the steady state problem of compression of an infinitely long strip by moving punches. In the first paper [4] which was concerned with a single moving punch, Fourier integrals were employed whereas the other study [5] where an infinite number of punches were considered, Fourier series were used. For both cases final results were based upon numerical solutions of

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Fredholm integral equations of the second kind. It appears that no such integral equation formulation has been attempted for the mixed boundary value problems of a rectangle considered in this study.

Although the type of eigenfunctions considered here was introduced in two dimensional problems by Papkovich [6] and Fadle [7] thirty years ago, its application remained limited until recently. Among several studies of stresses and displacements due to known forces, reference may be cited to Gaydon and Shepherd [8] and Johnson and Little [9], where attention was focussed on an approximate solution by a finite number of terms obtained after truncating the infinite series. We like to emphasize here that these studies may be satisfactory only in those cases where singularities in stresses due to geometrical discontinuities do not exist. In the presence of such singularities a mere truncation of the series is not permitted. For a correct solution the analysis will have to include the entire series which may be accomplished by adopting the procedure of the present study.

2. STATEMENT OF THE PROBLEM

The rectangular domain is referred by the two co-ordinate axes hx and hz. The bounding surfaces, $z = \pm 1$, are considered to be free from tractions. On the remaining bounding planes $x = \pm a$, the displacement u_x and the normal stress σ_{xx} are prescribed in the following manner.

$$u_x(a, z) = -(1-\nu)F_1(z); \quad 0 \le |z| \le b < 1 \tag{1}$$

$$\sigma_{xx}(a, z) = \frac{P_1(z)}{2}; \quad b < |z| \le 1 \tag{2}$$

$$u_x(-a, z) = (1-\nu)F_2(z); \quad 0 \le |z| \le c < 1 \tag{3}$$

$$\sigma_{xx}(-a, z) = \frac{P_2(z)}{2}; \quad c < |z| \le 1. \tag{4}$$

The units of length and stresses are chosen to be h and $2G$, respectively and F_1, F_2, P_1, P_2 are assumed to be even in z. We will also assume

$$\sigma_{xz}(\pm a, z) = 0 \quad 0 \le |z| \le 1. \tag{5}$$

It may be noted that the solution proposed in Section 3 is capable of considering shear stress distribution which is antisymmetric in z on the planes $x = \pm a$. However, the assumption of zero shear stress (equation 5) considerably simplifies the analysis and may be appropriate in several practical applications.

In the next two sections we will reduce the plane problem of elasticity of the rectangle with its edges $z = \pm 1$ stress free and subjected to the mixed boundary conditions (1–5) on the edges $x = \pm a$, to the solution of a coupled pair of Fredholm equations of the second kind. In Section 5 we will discuss results based upon a numerical solution of the Fredholm equations in the special case when the rectangle is compressed by two smooth rigid punches of identical shape on the surfaces $x = \pm a$.

3. SOLUTION OF NAVIER'S EQUATION

The displacements u_x and u_z are assumed in the following form

$$u_x = -(1-\nu)(Dx+E)+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\{A_n\sinh\lambda_n x+B_n\cosh\lambda_n x\}\lambda_n f_{1n}(z)$$

$$u_z = \nu Dz+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\{A_n\cosh\lambda_n x+G_n\sinh\lambda_n x\}\{f'_{1n}(z)+2(1-\nu)f'_{2n}(z)\} \tag{6}$$

where ν is the Poisson's ratio D and E are unknown real constants and A_n, B_n $(n = -\infty, \ldots -1, 1, \ldots \infty)$ are a set of complex constants. f_{1n} and f_{2n} constitute a set of complex valued eigenfunctions symmetric in z satisfying the differential equations

$$f''_{1n}+\lambda_n^2 f_{1n} = \lambda_n^2 f_{2n}; \quad f''_{2n}+\lambda_n^2 f_{2n} = 0 \tag{7a}$$

with the boundary conditions

$$(1-\nu)f_{2n}(\pm 1)+f_{1n}(\pm 1) = (1-\nu)f'_{2n}(\pm 1)+f'_{1n}(\pm 1) = 0. \tag{7b}$$

In the expressions (6, 7a, 7b) $f' = \mathrm{d}f/\mathrm{d}z$, λ_n $(n = 1, 2, \ldots \infty)$ are the nonzero roots of $\sin 2\lambda+2\lambda = 0$ in the first quadrant of the complex plane and $\lambda_{-n} = \bar{\lambda}_n$ $(n = 1, 2, \ldots \infty)$ where $\bar{\lambda}$ is the conjugate of λ. We have the following expressions for f_{1n} and f_{2n}.

$$f_{1n}(z) = \lambda_n\tan\lambda_n+2(1-\nu)\cos\lambda_n z-\lambda_n z\sin\lambda_n z$$
$$f_{2n}(z) = -2\cos\lambda_n z. \tag{8}$$

Associated with (8) there exists a set of functions g_{1k} and g_{2k} given by

$$4\cos^2\bar{\lambda}_k g_{1k}(z) = -(\bar{\lambda}_k\tan\bar{\lambda}_k+2\nu)\cos\bar{\lambda}_k z+\bar{\lambda}_k z\sin\bar{\lambda}_k z$$
$$4\cos^2\bar{\lambda}_k g_{2k}(z) = 2\cos\bar{\lambda}_k z \tag{9}$$

such that the following generalized orthonormality relation holds, see Ref. [10].

$$\int_{-1}^{1}\{f_{1n}(z)\overline{g_{2k}(z)}+f_{2n}(z)\overline{g_{1k}(z)}\} = 0; \quad k\neq n$$
$$= 1; \quad k = n. \tag{10a}$$

We also note that

$$\int_{-1}^{1}\{\nu f_{2n}(z)-f_{1n}(z)\}\,\mathrm{d}z = 0. \tag{10b}$$

Equations (6) yield the following expressions for the stresses σ_{xx}, σ_{xz} and σ_{zz}.

$$\sigma_{xx} = -D+\sum_n(A_n\cosh\lambda_n x+B_n\sinh\lambda_n x)\lambda_n^2\{f_{1n}(z)-\nu f_{2n}(z)\} \tag{11}$$

$$\sigma_{xz} = \sum_n(A_n\sinh\lambda_n x+B_n\cosh\lambda_n x)\lambda_n\{f'_{1n}(z)+(1-\nu)f'_{2n}(z)\} \tag{12}$$

$$\sigma_{zz} = -\sum_n(A_n\cosh\lambda_n x+B_n\sinh\lambda_n x)\lambda_n^2\{f_{1n}(z)+(1-\nu)f_{2n}(z)\} \tag{13}$$

where $\sum_n$ indicates summation over $n = -\infty, \ldots -1, 1, \ldots \infty$.

With the help of (7a) it may be easily shown that expressions (6) satisfy the Navier's equation and equations (12, 13, 7b) imply that the surfaces $z = \pm 1$ of the rectangle are traction free.

Equations (1–5 and 6, 11, 12) yield the following system of series relations for the determination of the constants D, E, A_n and B_n ($n = -\infty, \ldots -1, 1, \ldots \infty$).

$$-(1-\nu)(Da+E)+\sum_n C_n^1\lambda_n f_{1n}(z) = -(1-\nu)F_1(z); \quad 0 \le |z| \le b \tag{14}$$

$$-D+\sum_n (A_n \cosh \lambda_n a + B_n \sinh \lambda_n a)\lambda_n^2\{f_{1n}(z)-\nu f_{2n}(z)\} = \frac{P_1(z)}{2}; \quad b < |z| \le 1 \tag{15}$$

$$-(1-\nu)(-Da+E)+\sum_n C_n^2\lambda_n f_{1n}(z) = (1-\nu)F_2(z); \quad 0 \le |z| \le c \tag{16}$$

$$-D+\sum_n (A_n \cosh \lambda_n a - B_n \sinh \lambda_n a)\lambda_n^2\{f_{1n}(z)-\nu f_{2n}(z)\} = \frac{P_2(z)}{2}; \quad c < |z| \le 1 \tag{17}$$

$$\sum_n C_n^1\lambda_n\{f_{1n}(z)-(1-\nu)f_{2n}(z)\} = 0; \quad 0 \le |z| \le 1 \tag{18}$$

$$\sum_n C_n^2\lambda_n\{f_{1n}(z)-(1-\nu)f_{2n}(z)\} = 0; \quad 0 \le |z| \le 1 \tag{19}$$

where

$$C_n^1 = A_n \sinh \lambda_n a + B_n \cosh \lambda_n a \tag{20}$$

and

$$C_n^2 = -A_n \sinh \lambda_n a + B_n \cosh \lambda_n a. \tag{21}$$

It may be noted that to obtain (18 and 19) we substituted (12) in (5) and integrated the resulting equations with respect to z. It is assumed that the order of integration and summation may be interchanged.

4. THE FREDHOLM EQUATIONS

To obtain the unknown constants from (14–19) we introduce two unknown functions $\phi_1(t_1)$, $b \le t_1 \le 1$ and $\phi_2(t_2)$, $c \le t_2 \le 1$ such that the displacements $u_x(-a, z)$, $c \le |z| \le 1$ and $u_x(a, z)$, $b \le z \le 1$ are given by

$$u_x(a, z) = -(1-\nu)(Da+E)+\sum_n C_n^1\lambda_n f_{1n}(z) = -(1-\nu)\left\{\int_b^{|z|} \phi_1(t)\,dt + F_1(b)\right\}; \quad b \le |z| \le 1$$

$$u_x(-a, z) = -(1-\nu)(-Da+E)+\sum_n C_n^2\lambda_n f_{1n}(z) = (1-\nu)\left\{\int_c^{|z|} \phi_2(t)\,dt + F_2(c)\right\}; \quad c \le |z| \le 1. \tag{22}$$

Making use of (14, 21, 18 and 10b) it may be easily shown by a permissible change in the order of integration and summation that

$$Da+E = (1-b)F_1(b)+\int_b^1 (1-t)\phi_1(t)\,dt+\int_0^b F_1(z)\,dz. \tag{23}$$

Similarly (16, 22, 19 and 10a) yield

$$Da - E = (1-c)F_2(c) + \int_c^1 (1-t)\phi_2(t)\,dt + \int_0^c F_2(z)\,dz. \tag{24}$$

The constants D and E may be evaluated by the use of (23 and 24) after $\phi_1(t)$ and $\phi_2(t)$ are determined. To obtain the Fredholm equations for the determination of $\phi_1(t)$ and $\phi_2(t)$ we employ the following procedure.

Equations (14, 21 and 18) in conjunction with the generalized orthonormality relation (10a) and the expressions (9) yield

$$2\cos^2\lambda_n C_n^1 \lambda_n = -\int_b^1 \phi_1(t)Q(\lambda_n, t)\,dt - \int_0^b F_1'(t)Q(\lambda_n, t)\,dt \tag{25}$$

where

$$Q(\lambda, t) = \cot\lambda \sin\lambda t - t\cos\lambda t. \tag{26}$$

It may be noted that we made use of a change of the order of integration in order to obtain (25).

A similar procedure when applied to equations (16, 22 and 19) gives the following expression for the complex constants C_n^2 $(n = -\infty, \ldots -1, 1, \ldots \infty)$

$$2\cos^2\lambda_n C_n^2 \lambda_n = \int_c^1 \phi_2(t)Q(\lambda_n, t)\,dt + \int_0^c F_2'(t)Q(\lambda_n, t)\,dt. \tag{27}$$

By the use of (25, 27 and 20) the constants A_n and B_n can now be expressed in terms $\phi_1(t)$, $\phi_2(t)$, $F_1'(t)$ and $F_2'(t)$. Substitution of these expressions and (8) in (11) and a change of the order of integration and summation yields the following expressions for the stress σ_{xx} at $x = \pm a$,

$$D + \sigma_{xx}(a, z) = \frac{d}{dz}\left[\int_b^1 \phi_1(t)L_1^0(t, z)\,dt + \int_0^b F_1'(t)L_1^0(t, z)\,dt + \int_c^1 \phi_2(t)L_2^0(t, z)\,dt + \int_0^c F_2'(t)L_2^0(t, z)\,dt\right] \tag{28}$$

$$D + \sigma_{xx}(-a, z) = \frac{d}{dz}\left[\int_c^1 \phi_2(t)L_1^0(t, z)\,dt + \int_0^c F_2'(t)L_1^0(t, z)\,dt + \int_b^1 \phi_1(t)L_2^0(t, z)\,dt + \int_0^b F_1'(t)L_2^0(t, z)\,dt\right] \tag{29}$$

where

$$\begin{aligned} L_1^0(t, z) &= \sum_{n=-\infty}^{\infty} \frac{\lambda_n \coth 2\lambda_n a Q(\lambda_n, t)Q(\lambda_n, z)}{2\cos^2\lambda_n} \\ L_2^0(t, z) &= \sum_{n=-\infty}^{\infty} \frac{\lambda_n \operatorname{cosech} 2\lambda_n a Q(\lambda_n, t)Q(\lambda_n, z)}{2\cos^2\lambda_n}. \end{aligned} \tag{30}$$

Let us now consider integration of the function

$$\frac{i\zeta \coth 2\zeta a}{\pi(\sin 2\zeta + 2\zeta)} Q(\zeta, t) Q(\zeta, z)$$

around the contour Γ consisting of the imaginary axis with indentations around the points $\zeta = (im\pi/2a)$ $(m = -M, \ldots -1, 0, 1, \ldots M)$, the semicircle $|\zeta| = (N+\frac{1}{2})\pi$ in the right hand half plane and the circles around the points $\zeta = \lambda_n$ $(n = -N, \ldots -1, 1, \ldots N)$ as well as $\zeta = m\pi$ $(m = 1, 2, \ldots N)$; where M is the largest integer less than or equal to $(2N+1)a$. Noting that the residues of the function at $\zeta = \lambda_n$, $\zeta = m\pi$ and $\zeta = im\pi/2a$ are

$$\frac{i\lambda_n \coth 2\lambda_n a Q(\lambda_n, t) Q(\lambda_n, z)}{4\pi \cos^2 \lambda_n}, \qquad -\frac{i(\coth 2m\pi a + 2m\pi a \operatorname{cosech}^2 m\pi a) \sin m\pi t \sin m\pi z}{2m\pi^2}$$

and

$$\frac{i}{2a\pi} \frac{m\pi/2a}{[\sinh(m\pi/a) + (m\pi/a)]} R\left(\frac{m\pi}{2a}, t\right) R\left(\frac{m\pi}{2a}, z\right)$$

respectively, where

$$R(\lambda, t) = \coth \lambda \sinh \lambda t - t \cosh \lambda t \tag{31}$$

we obtain

$$L_1^0(t, z) = L_3^0(t, z) - L_4^0(t, z) + L_5^0(t, z) \tag{32a}$$

where

$$L_3^0(t, z) = \sum_{m=1,2}^{\infty} \frac{(\coth 2m\pi a - 1 + 2m\pi a \operatorname{cosech}^2 m\pi a) \sin m\pi t \sin m\pi z}{m\pi} \tag{32b}$$

$$L_4^0(t, z) = \sum_{m=1,2}^{\infty} \frac{\frac{m\pi}{2a^2} R\left(\frac{m\pi}{2a}, t\right) R\left(\frac{m\pi}{2a}, z\right)}{\sinh \frac{m\pi}{a} + \frac{m\pi}{a}} \tag{32c}$$

$$L_5^0(t, z) = \sum_{m=1,2}^{\infty} \frac{\sin m\pi t \sin m\pi z}{m\pi}. \tag{32d}$$

In a similar manner by integrating the function

$$\frac{i\zeta \operatorname{cosech} 2\zeta a Q(\zeta, t) Q(\zeta, t)}{(\sin 2\zeta + 2\zeta)}$$

around the contour Γ it may be shown that

$$L_2^0(t, z) = L_6^0(t, z) - L_7^0(t, z) \tag{33a}$$

$$L_6^0(t, z) = \sum_{m=1,2}^{\infty} \frac{\operatorname{cosech} 2m\pi a(1 + 2m\pi a \coth 2m\pi a) \sin m\pi t \sin m\pi z}{m\pi} \tag{33b}$$

$$L_7^0(t, z) = \sum_{m=1,2}^{\infty} \frac{\frac{m\pi}{2a^2} \sec m\pi R\left(\frac{m\pi}{2a}, t\right) R\left(\frac{m\pi}{2a}, z\right)}{\sinh \frac{m\pi}{a} + \frac{m\pi}{a}}. \tag{33c}$$

With the help of the identities (32a and 33a) equation (28) reduces to

$$D+\sigma_{xx}(a,z) = \frac{d}{dz}\left\{\int_b^1 \phi_1(t)[L_3^0(t,z)-L_4^0(t,z)+L_5^0(t,z)]\,dt + \int_0^b F_1'(t)[L_3^0(t,z)-L_4^0(t,z)+L_5^0(t,z)]\,dt + \int_c^1 \phi_2(t)[L_6^0(t,z)-L_7^0(t,z)]\,dt + \int_0^c F_2'(t)[L_6^0(t,z)-L_7^0(t,z)]\,dt\right\}. \quad (34)$$

A similar equation may be obtained from (29).

We now introduce the following transformations

$$t = 1-u, \qquad z = 1-v, \qquad d = 1-b, \qquad e = 1-c \quad (35)$$

and denote

$$\phi_1(t) = \psi_1(u), \qquad \phi_2(t) = \psi_2(u)$$
$$F_1'(t) = F_3(u), \qquad F_2'(t) = F_4(u) \quad (36)$$
$$P_1(z) = P_3(v) \quad \text{and} \quad P_2(z) = P_4(v).$$

By use of (35 and 36) we can write (34) as

$$D+\sigma_{xx}(a,v) = -\frac{d}{dv}\left\{\int_0^d \psi_1(u)[L_5^0(u,v)+L_3^0(u,v)-L_8^0(u,v)]\,du + \int_d^1 F_3(u)[L_5^0(u,v)+L_3^0(u,v)-L_8^0(u,v)]\,du + \int_0^e \psi_2(u)[L_6^0(u,v)-L_9^0(u,v)]\,du + \int_e^1 F_4(u)[L_6^0(u,v)-L_9^0(u,v)]\,du\right\} \quad (37)$$

where

$$L_8^0(u,v) = \sum_{m=1,2}^{\infty} \frac{\frac{m\pi}{2a^2} R_1\left(\frac{m\pi}{2a},u\right) R_1\left(\frac{m\pi}{2a},v\right)}{\sinh\frac{m\pi}{a}+\frac{m\pi}{a}}$$

and (38)

$$L_9^0(u,v) = \sum_{m=1,2}^{\infty} \frac{\frac{m\pi}{2a^2} \sec m\pi R_1\left(\frac{m\pi}{2a},u\right) R_1\left(\frac{m\pi}{2a},v\right)}{\sinh\frac{m\pi}{a}+\frac{m\pi}{a}}$$

$R_1(\lambda, u)$ being given by

$$R_1(\lambda,u) = u\cosh\lambda(1-u) - \operatorname{cosech}\lambda\,\sinh\lambda u. \quad (39)$$

We now introduce two more unknown functions $\theta_1(u_1) 0 \leq u_1 \leq d$ and $\theta_2(u_2), 0 \leq u_2 \leq e$ such that

$$\psi_1(u) = -\frac{d}{du}\int_u^d \frac{\theta_2(y)}{\sqrt{(y^2-u^2)}}\,dy$$

and (40)

$$\psi_2(u) = -\frac{d}{du}\int_u^e \frac{\theta_2(y)}{\sqrt{(y^2-u^2)}}\,dy.$$

Substituting (40 in 37) and making use of equations (32b, d, 33b and 38) as well as some well known identities [11] we obtain

$$D+\sigma_{xx}(a, v) = -\frac{d}{dv}\left\{\frac{\pi}{2}\int_0^d \theta_1(y) \sum_{m=1,2}^{\infty} J_0(m\pi y)\sin mv\,dy\right\} + H^0(v) \tag{41}$$

where

$$\begin{aligned} H^0(v) = &-\frac{d}{dv}\left\{\frac{\pi}{2}\int_0^d \theta_1(y)[L_{10}^0(y,v)-L_{11}^0(y,v)]\,dy \right. \\ &+\int_d^1 F_3(u)[L_5^0(u,v)+L_3^0(u,v)-L_8^0(u,v)]\,du \\ &\left.+\frac{\pi}{2}\int_0^e \theta_2(y)[L_{12}^0(y,v)-L_{13}^0(y,v)]\,dy+\int_e^1 F_4(u)[L_6^0(u,v)-L_9^0(u,v)]\,du\right\}. \end{aligned} \tag{42}$$

L_{10}^0, L_{11}^0, L_{12}^0 and L_{13}^0 are expressed by the following,

$$L_{10}^0(y,v) = \sum_{m=1,2}^{\infty} (\coth 2m\pi a - 1 + 2m\pi a\,\mathrm{cosech}^2\, 2m\pi a) J_0(m\pi y)\sin m\pi v$$

$$L_{11}^0(y,v) = \frac{1}{a}\sum_{m=1,2}^{\infty} \frac{\frac{m\pi}{2a} R_3\left(\frac{m\pi}{2a}, y\right) R_1\left(\frac{m\pi}{2a}, v\right)}{\sinh\frac{m\pi}{a}+\frac{m\pi}{a}}$$

$$L_{12}^0(y,v) = \sum_{m=1,2}^{\infty} \mathrm{cosech}\, 2m\pi a(1+2m\pi a\coth 2m\pi a) J_0(m\pi y)\sin m\pi v$$

$$L_{13}^0(y,v) = \frac{1}{a}\sum_{m=1,2}^{\infty} \frac{\frac{m\pi}{2a}\sec m\pi R_3\left(\frac{m\pi}{2a}, y\right) R_1\left(\frac{m\pi}{2a}, v\right)}{\sinh\frac{m\pi}{a}+\frac{m\pi}{a}}$$

where

$$\begin{aligned} R_3(\lambda, y) = &(\cosh\lambda - \lambda\,\mathrm{cosech}\,\lambda) I_0(\lambda y) + \lambda y\cosh\lambda I_1(\lambda y) - \sinh\lambda L_0(\lambda y) \\ &-\lambda y \sinh\lambda\left[\frac{2}{\pi}+L_1(\lambda k)\right] \end{aligned} \tag{44}$$

I_0 and I_1 are modified Bessel functions and L_0 and L_1 are modified Struve functions.

It may now be noted that equation (2 or 15) is satisfied (refer to equations 36 and 41) if

$$D+\frac{P_3(v)}{2} = -\frac{\mathrm{d}}{\mathrm{d}v}\left\{\int_0^d \theta_1(y) \sum_{m=1,2}^{\infty} J_0(m\pi y)\sin m\pi v\,\mathrm{d}y\right\}+H^0(v);\quad 0 \le v < d. \tag{45}$$

Use of a well known identity [12]

$$\pi\sum_{m=1,2}^{\infty} J_0(m\pi y)\sin m\pi v = \frac{H(v-y)}{\sqrt{(v^2-y^2)}}-\int_0^{\infty}\exp(-s)\,\mathrm{cosech}\,sI_0(sy)\sinh sv\,\mathrm{d}s \tag{46}$$

reduces (45) to an equation of the Abel type whose solution may be written as

$$\theta_1(y)+y\int_0^d \theta_1(y')[K_2(y,y')-K_1(y,y')]\,\mathrm{d}y'+y\int_0^e \theta_2(y')K_3(y,y')\,\mathrm{d}y'$$
$$= -2y[D+r_1(y)+r_2(y)];\quad 0 \le y \le d \tag{47}$$

where,

$$K_1(y,y') = \int_0^{\infty} s\exp(-s)I_0(sy)I_0(sy')\,\mathrm{d}s \tag{48a}$$

$$K_2(y,y') = \pi\left[\sum_{m=1,2}^{\infty} m\pi(\coth 2m\pi a-1+2m\pi a\,\mathrm{cosech}^2\,2m\pi a)J_0(m\pi y)J_0(m\pi y')\right.$$
$$\left.-\frac{1}{a}\sum_{m=1,2}^{\infty}\frac{\frac{m\pi}{2a}R_3\left(\frac{m\pi}{2a},y\right)R_3\left(\frac{m\pi}{2a},y'\right)}{\sinh\frac{m\pi}{a}+\frac{m\pi}{a}}\right] \tag{48b}$$

$$K_3(y,y') = \pi\left[\sum_{m=1,2}^{\infty} m\pi\,\mathrm{cosech}\,2m\pi a(1+2m\pi a\coth 2m\pi a)J_0(m\pi y)J_0(m\pi y')\right.$$
$$\left.-\frac{1}{a}\sum_{m=1,2}^{\infty}\frac{\frac{m\pi}{2a}\sec m\pi R_3\left(\frac{m\pi}{2a},y\right)R_3\left(\frac{m\pi}{2a},y'\right)}{\sinh\frac{m\pi}{a}+\frac{m\pi}{a}}\right] \tag{48c}$$

$$r_1(y) = \frac{1}{\pi}\int_d^1\frac{F_3(u)}{\sqrt{(u^2-y^2)}}\,\mathrm{d}u-\frac{1}{\pi}\int_d^1 F_3(u)L_{14}^0(y,u)\,\mathrm{d}u$$
$$+\int_d^1 F_3(u)[L_{10}^0(y,u)-L_{11}^0(y,u)]\,\mathrm{d}u+\int_e^1 F_4(u)[L_{12}^0(y,u)-L_{13}^0(y,u)]\,\mathrm{d}u \tag{48d}$$

$$r_2(y) = \frac{1}{\pi}\int_0^y\frac{P_3(v)}{\sqrt{(y^2-v^2)}}\,\mathrm{d}v \tag{48e}$$

$$L_{14}^0(y,u) = \int_0^{\infty}\exp(-s)\,\mathrm{cosech}\,sI_0(sy)\sinh su\,\mathrm{d}s \tag{48d}$$

and L_{10}^0, L_{11}^0, L_{12}^0, L_{13}^0 and R_3 are given by (43 and 44).

By the use of (29, 32, 33, 35, 36, 38 and 40), it may be shown in a similar manner that equation (4) or (17) is satisfied if

$$\theta_2(y)+y\int_0^e \theta_2(y')[K_2(y,y')-K_1(y,y')]\,dy'+y\int_0^d \theta_1(y')K_3(y,y')\,dy'$$
$$= -2y[D+r_3(y)+r_4(y)];\quad 0\le y\le e \tag{49}$$

where

$$r_3(y) = \frac{1}{\pi}\int_e^1 \frac{F_4(u)}{\sqrt{(u^2-y^2)}}\,du-\frac{1}{\pi}\int_e^1 F_4(u)L^0_{14}(y,u)\,du$$
$$+\int_e^1 F_4(u)[L^0_{10}(y,u)-L^0_{11}(y,u)]\,du+\int_d^1 F_3(u)[L^0_{12}(y,u)-L^0_{13}(k,u)]\,du \tag{50a}$$

and

$$r_4(y) = \frac{1}{\pi}\int_0^y \frac{P_4(v)}{\sqrt{(y^2-v^2)}}\,dv. \tag{50b}$$

It should be noted that the first term in the right hand sides of the equations (47 and 49) is $-2Dy$. By the use of (23, 24, 35, 36 and 40) the constants D and E may be expressed as

$$2Da = T_1+T_2+\frac{\pi}{2}\left\{\int_0^d \theta_1(y)\,dy+\int_0^e \theta_2(y)\,dy\right\} \tag{51a}$$

$$2E = T_1-T_2+\frac{\pi}{2}\left\{\int_0^d \theta_1(y)\,dy-\int_0^e \theta_2(y)\,dy\right\} \tag{51b}$$

where

$$T_1 = (1-b)F_1(b)+\int_0^b F_1(z)\,dz \tag{51c}$$

and

$$T_2 = (1-c)F_2(c)+\int_0^c F_2(z)\,dz. \tag{51d}$$

The functions $\theta_1(y)$ and $\theta_2(y)$ may now be evaluated by solving the simultaneous Fredholm equations of the second kind obtained by the substitution of the value of D from (51a) in (47 and 49). After these two functions are obtained the real constants D and E may be evaluated by the use of (51a, b). The complex constants C_n^1 and C_n^2 can be expressed in terms of the functions $\theta_1(y)$ and $\theta_2(y)$ by the use of the equations (25 and 27), the transformations (35, 36 and 40) and some well known identities [11]. For the sake of brevity we give the final results as

$$2\cos^2\lambda_n C_n^1\lambda_n = -\frac{\pi}{2}\int_0^d \theta_1(y)Q_1(\lambda_n,y)\,dy-\int_0^b F_1'(t)Q(\lambda_n,t)\,dt \tag{52a}$$

$$2\cos^2\lambda_n C_n^2\lambda_n = \frac{\pi}{2}\int_0^e \theta_2(y)Q_1(\lambda_n,y)\,dy+\int_0^c F_2'(t)Q(\lambda_n,t)\,dt \tag{52b}$$

where

$$Q_1(\lambda, y) = (\cos\lambda - \lambda \operatorname{cosec}\lambda)J_0(\lambda y) - \lambda y \cos\lambda J_1(\lambda y) + \sin\lambda H_0(\lambda y) + \lambda y \sin\lambda\left\{\frac{2}{\pi} - H_1(\lambda y)\right\} \tag{52c}$$

and $Q(\lambda, t)$ is given by (26). In (52c) H_0 and H_1 are Struve functions of complex arguments. Since the complex constants A_n and B_n may be evaluated by the use of (20 and 52) the mixed boundary value problem described in Section 2 may now be considered as formally solved. Based upon this formulation, in the next section we discuss some problems of practical interest in which the stress and displacement fields are even in x. When symmetry about z-axis is maintained we must have $c = b$ or $e = d$ and, therefore,

$$\begin{aligned} F_1(z) &= F_2(z) = F(z) \\ \theta_1(t) &= \theta_2(t) = \theta(t) \\ P_1(z) &= P_2(z) = P(z) \\ P_3(v) &= P_4(v) = q(v) = P(1-v) \\ F_3(u) &= F_4(u) = F_5(u) = F'(1-u). \end{aligned} \tag{53}$$

In view of (53) the two integral equations (47 and 49) reduce to

$$\theta(y) + y\int_0^d \theta(y')K(y, y')\,\mathrm{d}y' = -2y\left[D + r(y) + \frac{h(y)}{2}\right]; \quad 0 \le y \le d \tag{54}$$

where

$$K(y, y') = K_4(y, y') - K_5(y, y') - K_1(y, y') \tag{55a}$$

$$K_4(y, y') = \pi \sum_{m=1,2}^{\infty} m\pi(\coth m\pi a - 1 + m\pi a \operatorname{cosech}^2 m\pi a)J_0(m\pi y)J_0(m\pi y') \tag{55b}$$

$$K_5(y, y') = \frac{2\pi}{a}\sum_{m=1,2}^{\infty} \frac{\dfrac{m\pi}{a}R_3\left(\dfrac{m\pi}{a}, y\right)R_3\left(\dfrac{m\pi}{a}, y'\right)}{\sinh\dfrac{2m\pi}{a} + \dfrac{2m\pi}{a}} \tag{55c}$$

$$r(y) = \int_d^1 F_5(u)L^0(u, y)\,\mathrm{d}u; \quad L^0(u, y) = \frac{1}{\pi\sqrt{(u^2 - y^2)}} - L_0^0(u, y). \tag{55d}$$

$$L_0^0(u, y) = -\frac{1}{\pi}\int_0^\infty \exp(-s)\operatorname{cosech} s I_0(sy)\sinh su\,\mathrm{d}s - \frac{2}{a}\sum_{m=1,2}^{\infty}\frac{\dfrac{m\pi}{a}R_3\left(\dfrac{m\pi}{a}, y\right)R_1\left(\dfrac{m\pi}{a}, u\right)}{\sinh\dfrac{2m\pi}{a} + \dfrac{2m\pi}{a}} + \sum_{m=1,2}^{\infty}(\coth m\pi a - 1 + m\pi a \operatorname{cosech}^2 m\pi a)J_0(m\pi y)\sin m\pi u \tag{55e}$$

$$h(y) = \frac{2}{\pi}\int_0^y \frac{q(v)}{\sqrt{(y^2 - v^2)}}\,\mathrm{d}v \tag{55f}$$

and $K_1(y, y')$, $R_3(\lambda, y)$ and $R_1(\lambda, u)$ are given by (48a, 44 and 39), respectively.

We also note that equations (51) yield

$$Da = (1-b)F(b) + \int_0^b F(z)\,dz + \frac{\pi}{2}\int_0^d \theta(y)\,dy \tag{56}$$

and the constant E is equal to zero. Moreover, the complex constants B_n ($n = -\infty, \ldots -1, 1, \ldots \infty$) are equal to zero and the equations (20 and 52) yield

$$2\lambda_n \cos^2 \lambda_n A_n \sinh \lambda_n a = -\frac{\pi}{2}\int_0^d \theta(y) Q_1(\lambda_n, y)\,dy - \int_0^b F'(z) Q(\lambda_n, t)\,dt \tag{57}$$

Q_1 and Q being given by (52c and 26), respectively.

From equations (11 and 7) it is clear that

$$-h\int_{-1}^{1} 2G\sigma_{xx}(x, z)\,dz = 4GhD = P_x \tag{58}$$

where P_x is the total compressive force per unit width in the direction of the coordinate axis x on any plane x = constant. This result (58) will be used in the next section.

5. APPLICATIONS

Infinite row of parallel edge cracks in a strip

The problem of determining the stresses in the neighborhood of edge cracks placed perpendicular to the axis of an infinite elastic strip is of importance in Fracture Mechanics. The effect of two co-planar symmetrically placed edge cracks in a strip has been investigated in [1]. If an infinite number of such edge cracks are arrayed periodically along the axis of the strip, it is necessary to solve a mixed boundary value problem of a rectangle. If the crack faces are subjected to the same pressure distribution, this problem becomes a special case of what has been considered previously. For this purpose, we consider a rectangle bounded by any two adjacent planes on which the cracks are located ($x = \pm a$) and the planes $z = \pm 1$, and assume that the pressure on each of the crack faces is $Gp_1(z)$, $b < |z| \leq 1$. Then, we have

$$\begin{gathered} F(z) = 0; \qquad P(z) = -P_1(z); \qquad D = 0 \\ r(y) = 0 \quad \text{and} \quad h(y) = -\frac{2}{\pi}\int_0^y \frac{P_1(1-v)}{\sqrt{(y^2-v^2)}}\,dv = -h_1(y) \end{gathered} \tag{59}$$

in view of which the integral equation (54) reduces to the following Fredholm equation of the second kind.

$$\theta(y) + y\int_0^d \theta(y') K(y, y')\,dy' = yh_1(y). \tag{60}$$

Various quantities of physical interest such as stress intensity factor and crack energy may be easily expressed in terms of the function $\theta(y)$. They have been evaluated for various values of crack length and crack spacing from numerical solutions of equation (60). These results will be reported elsewhere.

Compression of the rectangle by two smooth rigid punches of identical shape

We will consider here two cases; (a) flat punch, (b) punch in the form of a smooth curve. For case (a) we have $F(z) = \Delta_2$, $\Delta_2 h$ being the penetration of the punch and, for case (b) we will take $F(z)$ in the following form

$$F(z) = \sum_{k=0,2}^{M} \alpha_k z^k \tag{61}$$

where the constants α_k $(k = 2, 4, \ldots M)$ describe the shape of the punch and α_0 is yet arbitrary. In this study we wish to give in some details numerical results for a flat punch and a parabolic punch for which $\alpha_k = 0$ when $k \geq 4$. We also note here that for the type of contact problems under consideration, $P(z) = q(v) = 0$ and, therefore, $h(y) = 0$. Further, $2bh$ is the length over which the punch is in contact with the boundary of the rectangle.

Case (*a*). *Flat punch.* In this case $r(y) = 0$ and substituting $\theta(y) = -2D\theta_4(y)$ the integral equation (54) reduces to

$$\theta_4(y) + y \int_0^d \theta_4(y')K(y, y')\,\mathrm{d}y' = y \tag{62}$$

and from (56) we have

$$\Delta_2 = D\left\{a + \pi \int_0^d \theta_4(y)\,\mathrm{d}y\right\}. \tag{63}$$

The "effective resistance" defined as the ratio of the resultant load P_x and the penetration is of practical interest which may be obtained from (58) and (63) as

$$\frac{P}{G\Delta_2 h} = \frac{4}{a + \pi \int_0^d \theta_4(y)\,\mathrm{d}y}. \tag{64}$$

The stresses and displacements may be evaluated by the use of the equations (6, 11–13) with $B_n = 0$, $n = -\infty, \ldots -1, 1, \ldots \infty$. The constant D is connected to Δ_2 by the relation (63) and the complex constants A_n may be obtained by the use of (57) with $\theta(y) = -2D\theta_4(y)$ and $F'(z) = 0$. It should, however, be noted that displacement consists of complex valued functions and the evaluation of A_n necessitates a computation of Struve functions of complex arguments. In order to circumvent this difficulty, therefore, it appears essential to express the displacement and stresses in a different form by means of real functions. This may be accomplished by a substitution of (57) in (6, 11–13) and the use of the calculus of residues. This procedure is similar to what has been employed in obtaining the identity (32a). After a lengthy manipulation it may be shown that the stress σ_{xx} is given by the following expression,

$$-\frac{\sigma_{xx}(x, v)}{2G\Delta_2} = \frac{1}{a + \pi \int_0^d \theta_4(y)\,\mathrm{d}y}\Bigg[1 - \pi \int_0^d \theta_4(y)\Bigg\{\sum_{m=1,2}^{\infty} m\pi q_{1m}(x) J_0(m\pi y)\cos m\pi v$$

$$-\sum_{m=1,2}^{\infty} q_{2m}(x) R_3\left(\frac{m\pi}{a}, y\right) R_1'\left(\frac{m\pi}{a}, v\right)\Bigg\}\mathrm{d}y\Bigg] \tag{65a}$$

where

$$q_{1m}(x) = \text{cosech}\, m\pi a[\cosh m\pi x - m\pi x \sinh m\pi x + m\pi a \coth m\pi a \cosh m\pi x] \quad (65b)$$

$$q_{2m}(x) = \frac{2}{a} \frac{\frac{m\pi}{a}\cos\frac{m\pi x}{a}}{\cos m\pi\left(\sinh\frac{2m\pi}{a}+\frac{2m\pi}{a}\right)} \quad (65c)$$

R_3 is given by (44) and $R_1'(\lambda, v)$ is the derivative of R_1, equation (39), with respect to v and is given by

$$R_1'(\lambda, v) = (\cosh \lambda - \lambda\, \text{cosech}\, \lambda) \cosh \lambda v - \sinh \lambda \sinh \lambda v$$
$$- \lambda v \sinh \lambda \cosh \lambda v + \lambda v \cosh \lambda \sinh \lambda v. \quad (65d)$$

Similar expressions may be obtained for other components of stresses and displacements. It may be shown that in this case the stresses are singular at the points $x = \pm a$, $z = \pm b$.

Case (b). Punch in the form of a smooth curve. From (61 and 53) we have

$$F'(z) = \sum_{k=2,4}^{M} k\alpha_k z^{k-1} \quad (66)$$

and

$$F_5(u) = \sum_{k=2,4}^{M} k\alpha_k (1-u)^{k-1}.$$

In order to facilitate the analysis, we write

$$\theta(y) = \theta_3(y) - 2D_1\theta_4(y) \quad (67)$$

such that $\theta_3(y)$ satisfies

$$\theta_3(y) + y\int_0^d \theta_3(y')K(y, y')\,dy' = -2yr(y). \quad (68)$$

Where $K(y, y')$ and $r(y)$ are the same as in (55). The constant α_0 is obtained by satisfying

$$(1-b)F(b) + \int_0^b F(z)\,dz + \frac{\pi}{2}\int_0^d \theta_3(y)\,dy = 0. \quad (69)$$

It is clear from (69, 58 and 56) that if we take $\theta(y)$ equal to $\theta_3(y)$, the resultant force transmitted by the punch is zero. However, the stress field is singular at $x = \pm a$, $z = \pm b$. The stress $\sigma_{xx}(a, v)$ has a singularity of the form

$$\frac{1}{2}\frac{\theta_3(d)}{\sqrt{(v^2-d^2)}}.$$

This singularity vanishes if we superpose the solution for the case of a flat punch, case (a) indicated by the second term in (67). $\theta_4(y)$ satisfies the equation (62) and the constant D_1 may be evaluated as

$$D_1 = \frac{\theta_3(d)}{2\theta_4(d)}. \quad (70)$$

The process of superposition as described above has been used in contact problems when they were attempted by the use of integral transforms or dual series relations [3, 4].

The total force P_x transmitted by the punch and the penetration Δh are

$$P_x = 4GhD_1 \tag{71a}$$

$$\Delta h = (\alpha_0 + \Delta_1)h \tag{71b}$$

where

$$\Delta_1 = D_1\left\{a + \pi\int_0^d \theta_4(y)\,\mathrm{d}y\right\} \tag{71c}$$

and α_0, as obtained by the use of (61 and 69), is given by

$$\alpha_0 = -\frac{\pi}{2}\int_0^d \theta_3(y)\,\mathrm{d}y - \sum_{k=2,4}^{M} \alpha_k b^k\left(1 - \frac{kb}{k+1}\right). \tag{71d}$$

The "effective resistance" is therefore

$$\frac{P}{G\Delta h} = \frac{4D_1}{-\frac{\pi}{2}\int_0^d \theta_3(y)\,\mathrm{d}y - \sum_{k=2,4}^{M}\alpha_k b^k\left(1 - \frac{kb}{k+1}\right) + D_1\left\{a + \pi\int_0^d \theta_4(y)\,\mathrm{d}y\right\}}. \tag{71e}$$

For a parabolic punch we take

$$F(z) = \alpha_0 - \frac{\delta}{2}z^2$$
$$F_5(u) = -\delta(1-u) \tag{72}$$

and substituting $\theta_3(y) = \delta\theta_5(y)$ and $D_1 = \delta D_2$ we have from (67)

$$\theta(y) = \delta\theta_6(y) = \delta[\theta_5(y) - 2D_2\theta_4(y)] \tag{73}$$

where $\theta_4(y)$ is the solution of (62) and $\theta_5(y)$ satisfies

$$\theta_5(y) + y\int_0^d \theta_5(y')K(y, y')\,\mathrm{d}y' = 2yr_2(y) \tag{74}$$

$$r_2(y) = \frac{1}{\pi}\left[\log\frac{1+\sqrt{(1-y^2)}}{\mathrm{d}+\sqrt{(d^2-y^2)}} + \sqrt{(d^2-y^2)} - \sqrt{(1-y^2)}\right] + \int_d^1 (1-u)L_0^0(u, y)\,\mathrm{d}u \tag{75}$$

and $L_0^0(u, y)$ is given by (55e). From (70) we have

$$D_2 = \frac{\theta_5(d)}{2\theta_4(d)}. \tag{76a}$$

Equations (71) yield

$$P_x = 4Gh\delta D_2 \tag{76b}$$

$$\frac{\Delta}{\delta} = \frac{\alpha_0 + \Delta_1}{\delta} \tag{76c}$$

$$\frac{\Delta_1}{\delta} = D_2\left\{a + \pi\int_0^d \theta_4(y)\,\mathrm{d}y\right\} \tag{76d}$$

and

$$\frac{\alpha_0}{\delta} = -\frac{\pi}{2}\int_0^d \theta_5(y)\,dy + \frac{b^2(3-2b)}{6} \tag{76e}$$

from which we obtain

$$\frac{P}{G\Delta h} = \frac{4D_2}{-\frac{\pi}{2}\int_0^d \theta_5(y)\,dy + \frac{b^2(3-2b)}{6} + D_2\left\{a + \pi\int_0^d \theta_4(y)\,dy\right\}}. \tag{76f}$$

Using a process similar to what has been used to obtain (65a) we can show that $\sigma_{xx}(x, v)$ may be expressed as

$$-\frac{\sigma_{xx}(x,v)}{2G\delta} = D_2 + \frac{\pi}{2}\int_0^d \theta_6(y)\left\{\sum_{m=1,2}^{\infty} m\pi q_{1m}(x)J_0(m\pi y)\cos m\pi v - \sum_{m=1,2}^{\infty} q_{2m}(x)R_3\left(\frac{m\pi}{a}, y\right)R_1'\left(\frac{m\pi}{a}, v\right)\right\}dy + \int_d^1 F_5(u)\left\{\sum_{m=1,2}^{\infty} q_{1m}(x)\sin m\pi u\cos m\pi v - \sum_{m=1,2}^{\infty} q_{2m}(x)R_1\left(\frac{m\pi}{a}, u\right)R_1'\left(\frac{m\pi}{a}, v\right)\right\}du \tag{77}$$

where $\theta_6(y)$ is related to $\theta_5(y)$ and $\theta_4(y)$ by (73) and q_{1m}, q_{2m}, R_3, R_1' and R_1 are obtained from (65b, c, 44, 65d and 39), respectively. The integral equations (62 and 74) were

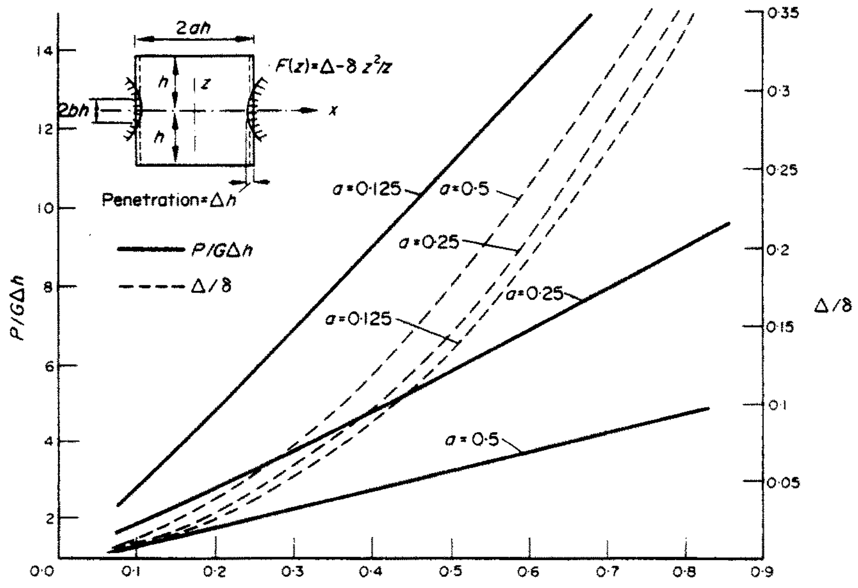

FIG. 1. Variation of $\frac{P}{G\Delta h}$ and $\frac{\Delta}{\delta}$ with b and a for a parabolic punch.

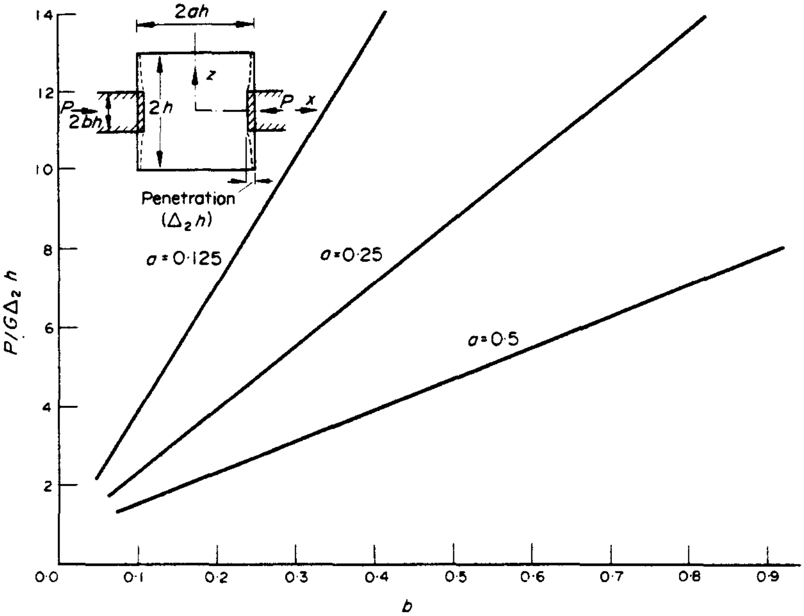

FIG. 2. Variation of $\frac{P}{G\Delta_2 h}$ with b and a for a flat punch.

numerically solved for various values of a and b ($d = 1-b$) by quadrature (trapezoidal rule) and the numerical values of Δ/δ and the effective resistance $P/G\Delta h$ are shown graphically in Fig. 1. The variation of $P/G\Delta_2 h$, the effective resistance for the case of a flat punch (case a), is shown in Fig. 2. The stress $\sigma_{xx}(x, z)$ is plotted against z (Fig. 3) for $x = 0$ and $x = 0{\cdot}9a$ for a flat punch as well as a parabolic punch (case b) when $a = 0{\cdot}25$ and $b = 0{\cdot}4$.

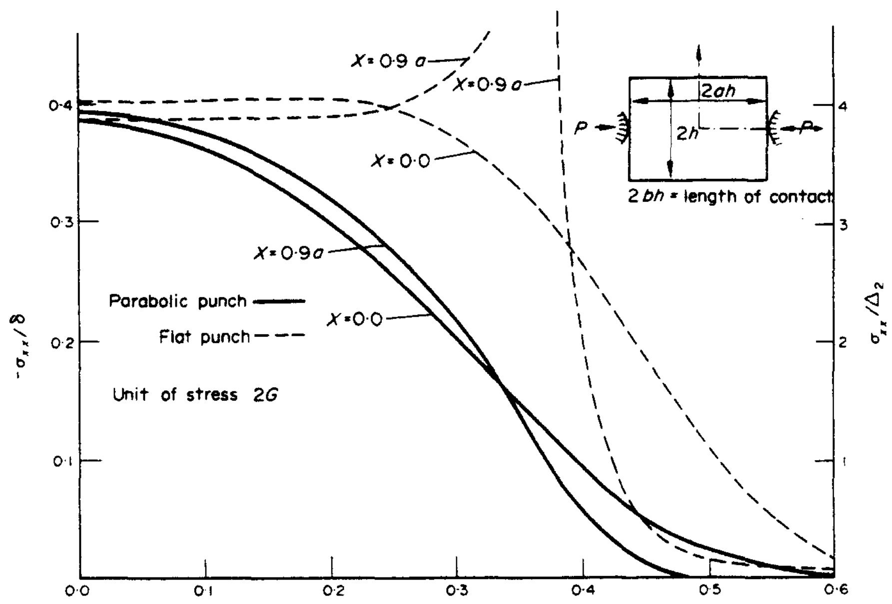

FIG. 3. Variation of nondimensional stress σ_{xx} for $b = 0{\cdot}4$, $a = 0{\cdot}25$.

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Абстракт—Дается формула, в виде системы двух интегральных уравнений Фредгольма второго рода, для обмего случая лрямоугольного упругого тела в плоском деформированном состоянии. Два параллельные края тела свободны от тягоьых усилий, но на осталвных краях предсказаны смешанные условия. Вид смешанных граничных условый может возникать вследствие сжатия двумя глад-кими штампами или некоторого периодиуеского расположения системы трещин в бесконеуно длинной полосе. Обсуждается подробно углубление прямоугольного тела плоским или параболическим штампом. Для величин, имеющих практический интерес, приводятся численные результаты.